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On Wilton's ripples: a special case of resonant interactions

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The phenomenon of second harmonic resonance for capillary-gravity waves is reconsidered here by the asymptotic method of multiple time and space scales. The periodic finite amplitude waves of permanent form found by Wilton in 1915 which correspond to this configuration are shown to be no more than a special case of the more general resonant interaction theory, and owe their existence to a critical choice of initial conditions. It is further suggested that the influence of viscous dissipation will render this solution virtually undetectable in a real liquid.

1. Introduction

In a remarkable paper, Simmons (1969) obtained a set of differential equations governing the dynamics of the resonant interaction of two wave modes that are harmonically related and identified the process as a 'second harmonic resonance'. This is no more than a special case of the now familiar second-order triad resonances for which two of the members of the triad coalesce, the closure being its second harmonic, and the propagation is uni-directional. The case investigated is easily seen to occur for capillary-gravity waves in virtue of the existence of a minimum value for the phase speed. That is, there is a wave-number such that a free wave and its free second harmonic can travel at identical phase speeds to the lowest order of approximation and it is natural to expect a resonance between these two free components. The fundamental wave-number corresponding to this possibility is $k = (g/2\gamma)^{\frac{1}{2}}$, g being gravity and γ the ratio of the surface tension to the density of the fluid. For water, this is a wavelength of about 2·4 cm.

The same configuration was investigated by Wilton (1915) who recognized the resonance as a secular forcing, and went on to eliminate the secularity by allowing the phase speed to be slightly shifted by an amount proportional to the small maximum slope of the waves, but retaining the assumption that the individual amplitudes remain constant. With proper allowance for what amounts to a 'Poincaré frequency shift',† Wilton obtained strictly periodic 'finite amplitude waves of permanent form' and further found that a necessary condition for this type of propagation is that the constant amplitudes of the fundamental and second harmonic be precisely in the ratio 2:1. The same configuration was examined again by Pierson & Fife (1961) with essentially the same results.

† This terminology and much of the subsequent is that repopularized by Cole (1968), where the multiple scale procedure to be used here is presented clearly.

McGoldrick (1965) suggested that this configuration could be analyzed in the framework of resonant interaction theory, but gave an algebraically wrong solution. Simmons, by a variational method, reinvestigated the second-order triad resonance theory with allowance for slow spatial and temporal modulations of the amplitudes of the modes. By suitable averaging of a Lagrangian for the system, he showed how to extract the equations for the slow variations of the lowest order amplitudes and discussed the properties of the three types of solutions that occur, namely: (i) pure amplitude modulations, (ii) amplitude modulations accompanied by frequency modulations and (iii) constant-amplitude linearly frequency modulated solutions which correspond to the Poincaré type solutions. For the degenerate case of second harmonic resonance, he correctly obtained the resonance equations and their type (i) solutions which have recently been verified experimentally by McGoldrick (1970), but Simmons apparently overlooked the type (ii) and (iii) solutions. It is the purpose of this brief paper to show that these other solutions exist and that the type (iii) solution is indeed the periodic permanent form solution of Wilton. We further suggest that the influence of inevitable viscous dissipation present at these scales of motion will make the detection of these periodic solutions practically impossible for ordinary liquids.

The elegant variational technique used by Simmons has the slight disadvantage that the higher order corrections cannot be obtained since the terms responsible for them are obliterated by the averaging procedure. It is for just this reason that the investigation briefly presented in the next section is begun *ab initio* by scaling the relevant dynamical equations in such a way that all dependent variables are $O(1)$ dimensionless quantitites and all constants (wave-numbers, frequencies) are made $O(1)$, the scaling parameter being, of course the maximum wave slope. Allowance for the slow-scale variations is made by requiring the dependent variables to contain explicit dependence on multiple time and space variables, and an asymptotically valid sequence of approximations, is obtained in the usual manner. The closure at the first-order results in Simmons's equations, but the formalism allows the higher order corrections to be determined sequentially, the first of which is briefly considered here.

2. The multiple scale procedure

If a prime is used to denote dimensional variables, then everything is made dimensionless according to the scheme $\zeta = \zeta'/a$, $\phi = \phi'/ac$, $\mathbf{x} = \kappa\mathbf{x}'$, $t = \Omega t'$, $k = k'/\kappa$, $\omega = \omega'/\Omega$ where a, c, κ and Ω are typical dimensional amplitude, phase speed, wave-number and frequency. Then the exact kinematic boundary condition at the free surface becomes

$$\zeta_t - \phi_z + \epsilon\nabla\phi\,.\,\nabla\zeta = 0 \quad \text{at} \quad z = \epsilon\zeta, \tag{2.1}$$

where $\epsilon = a\kappa$. The dynamic boundary condition is obtained from the substantial derivative of the Bernoulli equation evaluated at the free surface which is

$$\phi_{tt} + \lambda\phi_z - \mu DF(\zeta)/Dt + \epsilon(\nabla\phi\,.\,\nabla\phi)_t + \tfrac{1}{2}\epsilon^2[\nabla\phi\,.\,\nabla(\nabla\phi\,.\,\nabla\phi)] = 0 \tag{2.2}$$

at $z = \epsilon\zeta$, exactly. Here $\lambda = g\kappa/\Omega^2$, $\mu = \gamma\kappa^3/\Omega^2$, $D/Dt = \partial/\partial t + \epsilon\nabla\phi.\nabla$, γ is the surface tension coefficient divided by density, $F(\zeta)$ is the sum of the principal curvatures, and the atmospheric pressure is as usual taken to be constant. Equations (2.1) and (2.2) together with Laplace's equation and the condition that $\nabla\phi$ vanish as $z \to -\infty$ govern the problem exactly.

Now suppose that the dependent variables may be expanded according to

$$\zeta = \zeta^{(1)} + \epsilon\zeta^{(2)} + \ldots; \quad \phi = \phi^{(1)} + \epsilon\phi^{(2)} + \ldots. \tag{2.3}$$

Further, let the amplitudes of the motion be slowly varying functions of time and space, the dependence being formally on 'slow scales' $T = \epsilon t$ and $\mathbf{X} = \epsilon\mathbf{x}$. Then Taylor series expansion of the free surface conditions about $z = 0$ together with $\partial/\partial t \to \partial/\partial t + \epsilon\,\partial/\partial T$ and $\partial/\partial x_i \to \partial/\partial x_i + \epsilon\,\partial/\partial X_i$ yields a sequence of approximations for propagation in one direction, taken to be that of x. The $O(1)$ equations are

$$\left.\begin{aligned} &\zeta^{(1)}_t - \phi^{(1)}_z = 0 \quad (z = 0),\\ &\phi^{(1)}_{tt} + \lambda\phi^{(1)}_z - \mu\phi^{(1)}_{xxz} = 0 \quad (z = 0),\\ &\phi^{(1)}_{xx} + \phi^{(1)}_{zz} = 0,\\ &\phi^{(1)}_x \quad \text{and} \quad \phi^{(1)}_z \to 0 \quad (z \to -\infty).\end{aligned}\right\} \tag{2.4a–d}$$

These are of course the familiar equations of the linearized theory. In the second, the kinematic boundary condition has been used to eliminate $\zeta^{(1)}$ resulting in a linear equation for $\phi^{(1)}$ alone. The last three of (2.4) constitute the problem for the $O(1)$ potential on the rapidly varying scales, and the $O(1)$ free surface displacement is determined via the kinematic condition (2.4*a*). As is usual, the slow scale behaviour of the $O(1)$ solutions is arbitrary and cannot be determined until the next order problem is set, which follows.

The $O(\epsilon)$ equations then are, after some manipulation

$$\left.\begin{aligned} &\zeta^{(2)}_t - \phi^{(2)}_z = \phi^{(1)}_Z - \zeta^{(1)}_T - (\zeta^{(1)}\phi^{(1)}_x)_x,\\ &\phi^{(2)}_{tt} + \lambda\phi^{(2)}_z - \mu\phi^{(2)}_{xxz} = -\{2\phi^{(1)}_{tT} + \lambda\phi^{(1)}_Z - \mu\phi^{(1)}_{xxZ} - 2\mu\phi^{(1)}_{xXz}\}\\ &\qquad + 3\mu(\zeta^{(1)}_x\phi^{(1)}_{zz})_x - (\phi^{(1)}_x\phi^{(1)}_x + \phi^{(1)}_z\phi^{(1)}_z)_t,\\ &\phi^{(2)}_{xx} + \phi^{(2)}_{zz} = -2(\phi^{(1)}_{xX} + \phi^{(1)}_{zZ}),\\ &(\phi^{(2)}_x + \phi^{(1)}_X) \quad \text{and} \quad (\phi^{(2)}_z + \phi^{(1)}_Z) \to 0 \quad (z \to -\infty).\end{aligned}\right\} \tag{2.5a–d}$$

In order that the sequence of approximations (2.3) be bounded, the secularity producing terms on the right-hand side of (2.5*b*) if any must be eliminated, resulting in a differential equation for the slow scale behaviour of the $O(1)$ solutions.

For the problem considered here, we look for solutions that to $O(1)$ consist of a superposition of a fundamental wave and its second harmonic, both of which independently satisfy equations (2.4*a–d*). Writing

$$\zeta^{(1)} = \sum_{j=\pm1,\,\pm2} a_j(X,T)\,e^{ji\psi} \tag{2.6}$$

where $\psi = kx - \omega t$, the rapidly varying phase function, and $a_{-j} = a_j^*$ in order that $\zeta^{(1)}$ be real, then the $O(1)$ solutions are

$$\phi^{(1)} = -i\frac{\omega}{|k|}\sum_{j=\pm1,\,\pm2} b_j(X,Z,T)\,e^{|jk|z + ji\psi}, \tag{2.7}$$

where $b_{-j} = -b_j^*$. Equation (2.4b) yields the dispersion relation

$$(j\omega)^2/|jk| = \lambda + \mu(jk)^2 \quad (j = 1, 2) \tag{2.8}$$

and in order that both the fundamental and the second harmonic ($j = 1, 2$) satisfy (2.8), then we must have $k^2 = \lambda/2\mu$, or in dimensional terms $k' = (g/2\gamma)^{\frac{1}{2}}$. The kinematic condition (2.4a) yields no more than $b_j(X, O, T) = a_j(X, T)$.

Substitution of (2.6) and (2.7) into the $O(\epsilon)$ dynamic boundary condition (2.5b) yields after a bit of algebra

$$\begin{aligned}\phi_{tt}^{(2)} + \lambda\phi_z^{(2)} - \mu\phi_{xxz}^{(2)} = \Big[& 2\frac{\omega^2}{|k|}\{a_{1_T} + U_1 a_{1_X} + i\omega|k|\, a_1^* a_2\} e^{i\psi} \\ & + 4\frac{\omega^2}{|k|}\{a_{2_T} + U_2 a_{2_X} + i\omega|k|\, a_1^2/2\} e^{2i\psi} \\ & + 18 i\omega^3 a_1 a_2 e^{3i\psi} + 32 i\omega^3 a_2^2 e^{4i\psi}\Big] \\ & + [\ldots]^*. \end{aligned} \tag{2.9}$$

Here, $U_1 = 5\omega/6|k|$ and $U_2 = 7\omega/6|k|$ are the group velocities of the fundamental and second harmonic waves respectively. The condition that $\phi^{(2)}$ be free of secularities clearly is that the expressions in the curly brackets vanish separately, or

$$\left.\begin{aligned} a_{1_T} + U_1 a_{1_X} &= -i\omega|k|\, a_1^* a_2, \\ a_{2_T} + U_2 a_{2_X} &= -\tfrac{1}{2} i\omega|k|\, a_1^2. \end{aligned}\right\} \tag{2.10}$$

These are precisely the second harmonic resonance equations obtained by Simmons (1969), here expressed in scaled complex form. Writing $a_j = \frac{1}{2}A_j e^{i\theta_j}$, where A_j and θ_j are the real amplitudes and phases, then on separation of real and imaginary parts, (2.10) become

$$\left.\begin{aligned} \left(\frac{\partial}{\partial T} + U_1\frac{\partial}{\partial X}\right) A_1 &= -\frac{\omega|k|}{2}\sin\theta\, A_1 A_2, \\ \left(\frac{\partial}{\partial T} + U_2\frac{\partial}{\partial X}\right) A_2 &= +\frac{\omega|k|}{4}\sin\theta A_1^2, \\ A_1\left(\frac{\partial}{\partial T} + U_1\frac{\partial}{\partial X}\right)\theta_1 &= -\frac{\omega|k|}{2}\cos\theta\, A_1 A_2, \\ A_2\left(\frac{\partial}{\partial T} + U_2\frac{\partial}{\partial X}\right)\theta_2 &= -\frac{\omega|k|}{4}\cos\theta A_1^2, \end{aligned}\right\} \tag{2.11}$$

where $\theta = 2\theta_1 - \theta_2$ may be called the relative phase. There is no general solution of (2.11) subject to arbitrary initial conditions yet; however Simmons has obtained special solutions for the case where the partial differential operators may be expressed as total derivatives with respect to a single characteristic variable. For simplicity, however, and for ease of comparison with the Wilton-type solutions, we will suppose for the rest of this paper that there is no spatial variation of the amplitudes on the long scale, $\partial/\partial X = 0$. The desired features of the solutions are not lost.

The first two of (2.11) possess an 'energy integral' $A_1^2 + 2A_2^2 = E$ which is proportional to the $O(1)$ energy density in the system and is independent of the relative phase. Further, following Simmons, an integral involving the relative phase is easily obtained (divide the first of (2.11) by the third, the second by the fourth, rearrange, add, and integrate), being $A_1^2 A_2 \cos\theta = L$, and elementary considerations will show that $0 \leqslant L^2 \leqslant \frac{2}{27}E^3$. Then (2.11) become

$$\left.\begin{aligned} \frac{d}{dT}A_1^2 &= -\frac{\omega|k|}{\sqrt{2}}(-A_1^6+EA_1^4-2L^2)^{\frac{1}{2}},\\ \frac{d}{dT}A_2^2 &= \frac{\omega|k|}{2}(4A_2^6-4EA_2^4+E^2A_2^2-L^2)^{\frac{1}{2}},\\ A_1^2\frac{d\theta_1}{dT} &= 2A_2^2\frac{d\theta_2}{dT} = -\tfrac{1}{2}L\omega|k|. \end{aligned}\right\} \tag{2.12}$$

The first two of (2.12) integrate directly giving

$$\left.\begin{aligned} A_1^2(T) &= \beta_3-(\beta_3-\beta_2)\,\mathrm{sn}^2(\Xi;\hat{k}),\\ A_2^2(T) &= \tfrac{1}{2}\{\beta_1+\beta_2+(\beta_3-\beta_2)\,\mathrm{sn}^2(\Xi;\hat{k})\}, \end{aligned}\right\} \tag{2.13}$$

with $\Xi = [\frac{1}{2}(\beta_3-\beta_1)]^{\frac{1}{2}}\,\frac{1}{2}\omega|k|(T-T_0)$, and where the constants β are the roots of the cubic equation $Ex^2-x^3-2L^2=0$, and $-\frac{1}{3}E \leqslant \beta_1 \leqslant 0 \leqslant \beta_2 \leqslant \frac{2}{3}E \leqslant \beta_3 \leqslant E$, and the modulus is given by $\hat{k}^2 = (\beta_3-\beta_2)/(\beta_3-\beta_1)$. The individual phases then follow by integration of the remaining two of (2.12), or

$$\left.\begin{aligned} \theta_1(T)-\theta_1(T_0) &= -\sqrt{2}\,L\beta_3^{-1}(\beta_3-\beta_1)^{-\frac{1}{2}}\,\Pi(\Xi,\alpha_1^2,\hat{k}),\\ \theta_2(T)-\theta_2(T_0) &= -\sqrt{2}\,L(\beta_1+\beta_2)^{-1}(\beta_3-\beta_1)^{-\frac{1}{2}}\,\Pi(\Xi,\alpha_2^2,\hat{k}), \end{aligned}\right\} \tag{2.14}$$

where Π is the incomplete elliptic integral of the third kind, with $\alpha_1^2 = 1-\beta_2/\beta_3$ and $-\alpha_2^2 = (\beta_3-\beta_2)/(\beta_1+\beta_2)$. The complete solution then consists of periodic amplitude modulations together with periodic zero-mean phase modulations (same period) superimposed on a slow linear growth of the phases. This linear growth is most easily interpreted as being an $O(\epsilon)$ frequency shift: that is, all second harmonic resonant configurations (save $L = 0$, to which we shall return) possess an $O(\epsilon)$ correction to their phase speeds.

A graphic presentation of the amplitude modulations, devoid of the cumbersome elliptic notation is most easily presented in the phase-planes for the individual amplitudes (figure 1) which are easily constructed from (2.12) directly. The outermost trajectories are those for which $L = 0$, for which the individual phases are constant and the relative phase is exactly $\pm\frac{1}{2}\pi$. The modulation period becomes infinite, and the solutions degenerate to their limiting cases of hyperbolic functions. Simmons concluded that all solutions for second harmonic resonant configurations coalesced to these, which is not so. The existence of this solution clearly requires very special initial conditions: namely, if the relative phase is ever $\pm\frac{1}{2}\pi$, then it remains so for all time. The spatially modulated counterparts of these solutions have been investigated in detail experimentally by McGoldrick (1970) with particular attention to the initial conditions, and need not be further treated here.

For initial conditions such that $0 < L^2 < \frac{2}{27}E^3$ which corresponds to the inner trajectories in figure 1, the picture is one of simultaneous amplitude and phase modulations. The individual amplitudes can never vanish, and the phases are frequency modulated about a slightly shifted mean frequency. The relative phase θ modulates periodically in the range $-\frac{1}{2}\pi < -\theta_m \leqslant \theta \leqslant +\theta_m < \frac{1}{2}\pi$ where $\theta_m = \cos^{-1}[L/(\frac{2}{27}E^3)^{\frac{1}{2}}]$. The phase difference is maximum when the amplitudes are undergoing their greatest growth, which occurs when $A_1^2 = 4A_2^2$ indicated by the dashed vertical lines in figure 1, and the waves are in phase ($\theta = 0$) at times when the amplitude changes are zero.

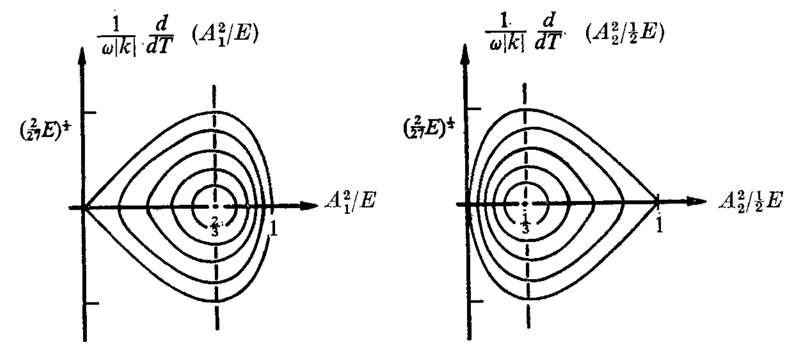

FIGURE 1. Phase plane trajectories of the individual amplitudes. The outermost trajectory is for $L = 0$ and for which the phase modulations disappear. The innermost trajectory (points) correspond to $L^2 = \frac{2}{27}E^3$, for which the amplitude modulations disappear and the phase changes are linear. The maximum relative phase θ_m occurs at the intersections of the trajectories with the dashed lines.

The remaining possibility is for initial conditions such that $L = (\frac{2}{27}E^3)^{\frac{1}{2}}$. For non-zero amplitudes this requires that $\cos\theta = 1$, whence the relative phase modulation vanishes. The trajectories become the single points on the abscissae of figure 1; that is, the amplitude modulations vanish entirely. The solutions become $A_1^2 = 4A_2^2 = \frac{2}{3}E$, their initial values, and the individual phases are linearly modulated, being $\theta_1 = \mp\frac{1}{4}A_1\omega|k|\,T$ and $\theta_2 = \mp\frac{1}{2}A_1\omega|k|\,T$, the upper or lower sign chosen according to the sign of the amplitudes, $A_2 = \pm\frac{1}{2}A_1$, and we have chosen $\theta_i(T_0) = 0$ for simplicity. For this case, the complete $O(1)$ solution for ζ is

$$\zeta^{(1)} = A_1\cos(kx-\omega t\mp\tfrac{1}{4}\omega|k|\,A_1T)\pm\tfrac{1}{2}A_1\cos 2(kx-\omega t\mp\tfrac{1}{4}\omega|k|\,A_1T),$$

or in dimensional terms

$$\zeta^{(1)'} = A_1'\{\cos[k'x'-\omega'(1\pm\tfrac{1}{4}A_1'k')\,t']\pm\tfrac{1}{2}\cos 2[k'x'-\omega'(1\pm\tfrac{1}{4}A_1'k')\,t']\}. \quad (2.15)$$

This is precisely the lowest order periodic solution found by Wilton. The upper signs correspond to a 'gravity-type' profile with phase speed increased by an amount proportional to the *first power* of the wave slope, the lower sign combinations are of 'capillary-type' with a decreased speed. It is remarkable that the existence of these 'periodic solutions of finite amplitude' require very special initial conditions, namely that the relative phase be zero *and* the initial amplitudes be exactly in the ratio $\pm\frac{1}{2}$. With any other choice of initial amplitudes, the intermediate 'everything modulating' case obtains, which is *not* strictly periodic.

3. The higher order corrections

Still retaining the simplifying assumption that $\partial/\partial X_j = 0$, it is a relatively simple matter to determine $\phi^{(2)}$ from (2.9) after removal of the secular forcing terms from the right-hand side, and then $\zeta^{(2)}$ from (2.5a). A particular solution of (2.9) is easily seen to be

$$\phi^{(2)\dagger} = [9i\omega a_1 a_2\, e^{3|k|z+3i\psi} + 4i\omega a_2^2\, e^{4|k|z+4i\psi}] + [\ldots]^*.$$

Then $\phi^{(2)}$ consists of this plus an arbitrary multiple of the solutions of the homogeneous part of (2.9) satisfying Laplace's equation, which may be taken to be

$$[i\omega\Delta_1\, e^{|k|z+i\psi} + i\omega\Delta_2\, e^{2|k|z+2i\psi}] + [\ldots]^*,$$

where Δ_1 and Δ_2 are arbitrary and depend on the initial condition as will be seen below. This solution for $\phi^{(2)}$ allows us to determine $\zeta^{(2)}$ from (2.5a), which is easily seen to be

$$|k|^{-1}\zeta^{(2)} = [(2a_1^* a_2 - \Delta_1)\, e^{i\psi} + (\tfrac{3}{4}a_1^2 - \Delta_2)\, e^{2i\psi} - 6a_1 a_2\, e^{3i\psi} - 2a_2^2\, e^{4i\psi}] + [\ldots]^*. \quad (3.1)$$

We wish now to compare the complete solution correct to $O(\epsilon)$ with those previously determined, namely those for which the initial amplitudes and phases are such that the amplitude modulations disappear altogether. That is, we specify that $A_2 = \pm\tfrac{1}{2}A_1$ and $\theta = 0$ at $T = T_0$ so that (in dimensional terms now)

$$\zeta' = A'\{[1 \pm \tfrac{1}{4}(2-d_1)\, A'k']\cos\chi \pm \tfrac{1}{2}[1 \pm (\tfrac{3}{4} - d_2)\, A'k']\cos 2\chi \mp \tfrac{3}{2}A'k'\cos 3\chi - \tfrac{1}{4}A'k'\cos 4\chi\}, \quad (3.2)$$

where $\chi = k'x' - \omega'(1 \pm \tfrac{1}{4}A'k')\, t'$, and the upper or lower signs correspond to gravity-type or capillary-type profiles respectively. The real constants d are $d_1 = |2\Delta_1/A^2|$ and $d_2 = |\Delta_2/A^2|$. The wave form is strictly periodic, and the phase speed is increased or decreased by an amount one-quarter of the maximum slope of the fundamental component. These initial conditions are the only ones which will produce a strictly periodic solution.

The special solutions (3.2) together with the phase speed corrections are the solutions found by Wilton and those determined subsequently by Pierson & Fife apart from their unimportant algebraic errors in the coefficients of the third- and fourth-harmonic terms, provided due account of the arbitrary constants d_1 and d_2 be taken. But this too is unimportant since in an experiment, the amplitude of the fundamental is determined by external conditions, and the amplitudes in (3.2), being arbitrary, must be chosen to coincide with the measured value. This indeed is the assumption used by Pierson & Fife, who set $d_1 = 2$ once and for all.

Before the edifying work of Simmons, the two types of solutions of this non-linear problem were the pure amplitude modulations described by McGoldrick which were measured in detail and the periodic solutions for which the system responded with a Poincaré frequency shift. It is perhaps ironic that those solutions are no more than those dictated by very special initial conditions within the framework of a much more general expansion scheme.

We further suggest that detection of the periodic solutions is an extremely difficult matter for a real liquid. The requirement for their existence is two-fold: that the amplitudes of the fundamental and second harmonic be precisely in the ratio $\pm 2:1$ and that they be in phase. But viscous attenuation will affect both components differently. That is, with appropriate initial conditions at a wave-maker, the spatial logarithmic decrements (to the lowest order) of the two components based on a scrupulously clean surface are $-2\nu k'^2/U_1'$ and $-8\nu k'^2/U_2'$ per unit length respectively, which are in a ratio of 7/20. Within a short distance of the wave-maker the necessary amplitude ratio will not be obtained and the interaction will produce modulations. This has been observed qualitatively in preliminary experiments, more or less, but any detailed measurements are too tedious for the reward and have been laid aside permanently.

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